Method for the Numerical Solution of Turbulent Flow Problems

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An integro-differential formulation, previously established for the study of incompressible laminar flow problems is modified and used in conjunction with the turbulent kinetic energy-energy dissipation rate model for the study of turbulent flow problems. It is demonstrated that the kinematics of the mean turbulent motion is identical to that of the laminar motion. The kinetic aspect of the turbulent flow is significantly more complex. The integro-differential formulation for the turbulent flow retains the important features of the laminar flow and permits the solution field to be confined to the vortical region. Numerical results are presented for several problems involving simple flow boundaries and compared with available experimental data.

Introduction

A FOCAL point of research in computational fluid dynamics has been, for more than two decades, the development of suitable numerical methods for the problem of general viscous flow containing appreciable separation. During the past few years, extensive progress has been made in the application of coordinate-transformation techniques, finite-difference methods, and finite-element methods to this problem.[1,2] Nevertheless, general viscous flows of practical importance today remain mostly beyond the scope of prevailing methods.

This limitation of numerical methods has long been known to be especially acute for external flows at high Reynolds numbers past finite solids of complex shapes. Because such flows are of far greater importance in application than are low Reynolds number flows involving simple boundary geometries, the limitation is of great concern to fluid dynamicists. With the longevity of this limitation, there has arisen a controversy regarding the practicality of predicting complex high Reynolds number general viscous flows using the computational approach in the foreseeable future.[3-6]

The limitation of prevailing methods is a direct consequence of at least three major obstacles: 1) excessive computer time and data storage needs for the solution of the equations of motion; 2) difficulties and uncertainties associated with the numerical treatment of certain boundary conditions; and 3) lack of accurate and general methods of describing turbulent transport phenomena.

These three obstacles have stood in tandem. In particular, the presence of the first and second obstacles have precluded the extensive numerical experimentation and calibration necessary for the establishment of suitable turbulence models for the general viscous flow. For this reason, although it has been widely recognized that the expressions "high Reynolds number separated flow" and "turbulent flow" are almost synonymous, the common objective of most of the relevant ongoing research activities has been the development of efficient and accurate algorithm for treating the equations of laminar motion and the associated boundary conditions. The realization of this objective is indeed a prerequisite for the development of a general computational capability for turbulent flows.

Several years ago, the first author of this article and his co-workers initiated a program of research with the initial goal of overcoming the first two obstacles just described for incompressible flow problems. The progress that has been made toward the attainment of this goal has been encouraging and a more ambitious goal of surmounting the third obstacle was recently introduced. In particular, an integro-differential approach, which has been established for the laminar incompressible flow problem, is modified, extended, and utilized in conjunction with a two-equation turbulence model for the time-dependent turbulent flow problem. The integro-differential approach is briefly reviewed in this article, with pertinent references made to the questions of solution efficiency and solution accuracy. The application of this approach to the study of turbulent flows is described. The solution procedure, developed on the basis of this approach, is discussed and illustrated by numerical results for several problems involving simple flow geometries.

Problem Formulation

Various aspects of the integro-differential approach have been described in a series of articles by the present authors and their co-workers.[7-15] Several unique and highly advantageous attributes of this approach have been demonstrated conclusively, by analyses and by numerical illustrations, for incompressible laminar flows. These attributes are retained by the approach when applied to turbulent flows.

The approach has its genesis in the obstacle of excessive computing needs for high Reynolds number flows past finite solids of complex shape. A common characteristic of such flows is that in the major part of the flowfield, the vorticity is negligible and the flow is essentially potential. The vorticity in the fluid in general must originate from the solid surfaces and spread into the fluid domain by diffusive and convective processes. Since diffusion proceeds at a much slower rate than does convection at high Reynolds numbers, the vorticity penetrates only a short distance into the fluid by diffusion before being carried away with the fluid. The vortical region, which may include attached boundary layers as well as separated flows, is therefore much smaller in extent than the potential region. Furthermore, the vortical region involves much larger gradients of velocity and other flow variables than does the potential region. The length scale appropriate to the vortical region is drastically smaller than that appropriate to the potential region. It is therefore difficult to devise a numerical grid that provides sufficient solution resolution and

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accuracy within the vortical region and yet does not contain a large number of data points in the potential region. Methods requiring the concurrent solution of both the vortical and the potential regions are therefore inefficient. For any given solid geometry, the length scale of the vortical region, particularly that of the boundary layers within it, decreases with increasing Reynolds number, while the length scale of the potential region remains essentially unchanged. As a consequence, the computing needs increase with increasing Reynolds number.

It is generally recognized that the aforementioned characteristics of high Reynolds number flows, which led to the "Reynolds number limitation" of computation methods, are common to the majority of laminar and turbulent, incompressible and compressible, time-dependent and steady, internal and external flows of practical interest. Flow problems in which this feature is not present do exist, but they are appropriately thought of as exceptions. The application of the integro-differential approach has been shown to be possible for the various types of flows just mentioned.[7-10] For incompressible flows, the approach permits the solution field to be confined to the vortical region only. This distinguishing feature of the approach removes the fundamental difficulty involved in the concurrent solution of the vortical and potential regions and therefore offers superior computational efficiency. For compressible flows, the integro-differential approach permits the solution field to be confined to the vortical region and the region where the expansion, or dilatation, is nonnegligible.[8] The implementation of the integro-differential approach, however, has thus far been concerned only with incompressible flows for which the vortical region is the only region of nonzero viscous force.

It needs to be emphasized that the ability of the integro-differential approach to confine the solution field has far-reaching implications. The significance of this approach in the analytical solution of viscous flow problems has not yet been fully explored. The advantage of this approach in the context of computational fluid dynamics is more evident to the present authors and forms the basis of the present article. It must also be emphasized that the ability of the integro-differential approach to confine the solution field is distinct from the notions of the boundary layer, the matched asymptotic expansion, or the triple-deck approaches. The latter approaches attempt to manage the concurrent solution of regions with vastly differing length scales by studying the regions separately. Since the regions "interact strongly" in flows with appreciable separation, the "matching" of the solutions in the various regions is fundamental to these latter approaches. With the integro-differential approach, a solution is sought only in the vortical region of the incompressible flow. The computation of the potential region of flow is made unnecessary by this new approach.

The above described cause of computational inefficiency has also motivated recent extensive efforts in developing coordinate-transformation techniques and finite-element methods. A major consequence of these efforts is the introduction of a degree of flexibility into the location of data points, so that the flow boundaries are more accurately represented in the data grid and the use of an "expanding" data grid, with the physical spacing between data points increasing with the distance from the solid boundary, is possible. With such expanding data grids, the number of data points needed for the potential region is reduced, particularly if analytical studies are utilized to give guidance in the choice of the coordinate system or of the finite-element grid. Excessive computing needs, however, have remained a serious difficulty.

The use of the integro-differential approach does not preclude the adaptation of either the coordinate transformation technique or the finite-element method. In fact, some of the more recent studies using the integro-differential approach have incorporated a coordinate transformation[11] and a finite-element method.[12] The inherent flexibility of solution procedure offered by the integro-differential approach is a notable attribute of the approach.[13] It is worthy of note, however, that the integro-differential approach differs in spirit from the coordinate transformation technique or the finite-element method in that, instead of minimizing the difficulty associated with the "necessity" of treating two flow regions with vastly different length scales, the integro-differential approach actually removes this necessity and treats only the vortical region of the flow. This distinguishing feature of the approach, previously established for laminar flows, goes over in a straightforward manner to turbulent flows.

Differential Equations of Motion

The continuity and the Reynolds momentum equations for the mean turbulent motion of an incompressible fluid with constant density ρ and constant kinematic viscosity ν, given in indicial notation for a right-handed Cartesian coordinate system, are well known:

$$\frac{\partial v_j}{\partial x_j} = 0 \tag{1}$$

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 v_i}{\partial x_j \partial x_j} - \frac{\partial \overline{v_i' v_j'}}{\partial x_j} \tag{2}$$

where p is the mean static pressure, v_i is the i component of the mean velocity, $-\rho\, \overline{v_i' v_j'}$ is the ij component of the Reynolds stress tensor, with the prime indicating the fluctuating part of a flow variable. Equation (2) is a time-dependent momentum equation for the mean motion and is meaningful if the time scale for the turbulent motion is much smaller than that for the mean motion.

With the exception of the additional Reynolds stress term in Eq. (2), the mean velocity of a turbulent flow and the instantaneous velocity of a laminar flow satisfy the same set of differential equations. Furthermore, the no-slip condition on the solid surfaces and the freestream condition for external flows, satisfied by the instantaneous velocity of a laminar flow, are also satisfied by the mean velocity of a turbulent flow. The similarity of the equations and conditions governing the instantaneous laminar motion and the mean turbulent motion suggests that the various methods developed for laminar flows are also useful for turbulent flows. In particular, the foregoing discussions concerning the vortical and potential regions suggest the introduction of the mean vorticity of the turbulent flow as a dependent variable of the problem, in order to separate the problem into its kinematic and kinetic aspects, as was done for the laminar flow. The experience gained in the integro-differential study of laminar flows further suggests that the kinematic equations be recast into an integral representation, while the kinetic equations be kept in its familiar differential form.

Kinematics—Integral Representation

The mean vorticity ω_i is defined by

$$\omega_i = \epsilon_{ijk} \frac{\partial v_k}{\partial x_j} \tag{3}$$

where ϵ_{ijk} is the alternating unit tensor that equals 1 if the subscripts ijk are in cyclic order 12312, equals -1 if they are in anticyclic order 32132, and is zero otherwise.

Equations (1) and (3), together with appropriate velocity boundary conditions, constitute the kinematic aspect of the problem. That is, they permit the determination of the mean velocity field v_i for any known mean vorticity field ω_i and vice versa. Since Eqs. (1) and (3) are identical to the continuity and vorticity definition equations for the instantaneous velocity and vorticity, the kinematics of the mean turbulent motion is identical to that of the laminar motion. Therefore, the integral representations for the velocity vector previously developed for the laminar flow[7] go over directly to the turbulent flow. A general form of the integral representation, written in indicial notation, is

$$v_i = -\frac{1}{A}\left\{\int_R \frac{\epsilon_{ijk}\omega_{0,j}(x_0-x)_k}{[(x_0-x)_m(x_0-x)_m]^{d/2}}\mathrm{d}R_0 + \oint_B \frac{v_{0,m}(x_0-x)_m n_{0,i} - v_{0,m}(x_0-x)_i n_{0,m}}{[(x_0-x)_m(x_0-x)_m]^{d/2}}\mathrm{d}B_0 - \oint_B \frac{v_{0,i}(x_0-x)_m n_{0,m}}{[(x_0-x)_m(x_0-x)_m]^{d/2}}\mathrm{d}B_0\right\} \tag{4}$$

where B is the boundary of R; 0 indicates that the variables and integrations are in the x_0 space, e.g., $\omega_{0,j} = \omega_j(x_{0,k})$, etc.; $n_{0,i}$ is the i component of the outward normal unit vector; $A=4$ and $d=3$ for three-dimensional problems, $A=2$ and $d=2$ for two-dimensional problems; and v_i is a function of the spatial and time coordinates.

Equation (4) constitutes the entirety of the kinematics of the problem and is completely equivalent to differential Eqs. (1) and (3), subject to the velocity conditions on the boundary of the region of interest. This equation is derived from Eqs. (1) and (3) by the use of a principal solution in the vector form, vector potential, and Green's theorem.[7] No simplifications or approximations other than those contained in differential Eqs. (1) and (3) have been introduced. The equation is valid for attached as well as separated flows in the potential as well as vortical region. It permits the velocity field v_i to be evaluated explicitly, point by point. That is, with any prescribed velocity boundary condition on B and known vorticity distribution in R, the right-hand side of Eq. (4) gives, by numerical quadrature, a value for v_i at any specific data point of interest. The computation of a velocity value at each data point is accomplished independently of velocity computations for other data points. This ability to evaluate the velocity field explicitly is absent in finite-difference methods and in finite-element methods based on the concepts of variational principle or weighted residuals. By virtue of this ability, the integral representation permits the solution field for velocity to be confined to any portion of the flowfield of interest. In particular, in the case of incompressible flow, it is advantageous to confine the solution field to the vortical region.

The second integral in Eq. (4) represents the contributions of the velocity boundary condition to the velocity field within the boundary. For the external flow problem with no-slip condition on solid surfaces, this surface integral can be evaluated analytically.[14] For the special case of a single solid undergoing rectilinear translation only, this integral gives simply $v_{\infty,i}$, the freestream velocity relative to the solid. For this special case, one has

$$v_i = -\frac{1}{A}\int_R \frac{\epsilon_{ijk}\omega_{0,j}(x_0-x)_k}{[(x_0-x)_m(x_0-x)_m]^{d/2}}\mathrm{d}R_0 + v_{\infty,i} \tag{5}$$

At any given finite instant of time, the vorticity is nonnegligible only within some finite distance from the solids. Consequently, Eq. (5) satisfies the freestream velocity condition at infinity. The use of Eq. (5) therefore removes the difficulty of "far-field boundary conditions," which is well known to researchers using finite-difference or finite-element methods to treat the external flow problem. Also, the application of Eq. (5) on solid boundaries, where the velocity is known, leads to integral equations that permit the accurate simulation of the process of vorticity generation and depletion on the boundaries and thereby removes the well-known difficulty of "extraneous boundary condition." In a recent article,[14] the advantages of the integro-differential approach related to the numerical boundary conditions are discussed more fully for laminar flows. It was concluded that the approach overcomes the second obstacle, associated with the boundary conditions, stated in the Introduction of this article. These advantages obviously are also available to turbulent flows, for which an accurate treatment of the extraneous boundary condition on solid surfaces is of critical importance.

Suppose the vortical region of the flow contains N data points. The numerical quadrature of the integral in Eq. (5) for the velocity at these N points then requires an operation count on the order of N^2. Since this is a large operation count, it has been suggested that the integro-differential method is inefficient. It should be emphasized that Eq. (5) can be and has been utilized in hybrid methods[11] or flowfield segmentation methods,[13] resulting in a very efficient computation procedure with small operation counts, involving a much smaller number of data points N then otherwise necessary (since the computation is confined to the vortical region). The development of the integro-differential approach has progressed through several stages, each of which has led to a substantial reduction in computing needs. The recent computing need for laminar flows is about 1/20th of that reported in Ref. 15.

Kinetics—Vorticity Transport Equation

By taking the curl of Eq. (2), one obtains the following equation describing the transport of the mean vorticity:

$$\frac{\partial\omega_i}{\partial t} = \omega_j\frac{\partial v_i}{\partial x_j} - v_j\frac{\partial\omega_i}{\partial x_j} + \epsilon_{ijk}\frac{\partial^2}{\partial x_j\partial x_m}(2\nu S_{km}) - \epsilon_{ijk}\frac{\partial^2}{\partial x_j\partial x_m}(\overline{v_k' v_m'}) \tag{6}$$

where

$$S_{km} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x_m} + \frac{\partial v_m}{\partial x_k}\right) \tag{7}$$

is the mean rate of strain.

The first term on the right-hand side of Eq. (6) can be rewritten as $\omega_i S_{ij}$. This term represents amplification and rotation of the vorticity vector by the mean strain rate and vanishes in two-dimensional flows. The second term represents the convective transport of the mean vorticity by the mean velocity. The third term is expressible as $\nu\partial^2\omega_i/\partial x_j\partial x_j$ and represents the viscous diffusion of the mean vorticity. The last term is often expressed in terms of correlations between the fluctuating velocity and the fluctuating vorticity.[16] In the present work, this term is expressed in terms of the Reynolds stress for convenience.

The right-hand side of Eq. (6) is obviously zero in a region where the instantaneous vorticity and its derivatives are zero. There is, therefore, no need to compute velocity values outside the vortical region. In fact, provided that the Reynolds stress distribution is known, the velocity field needs to be computed only in the region of nonnegligible mean vorticity. With this provision, the ability of the integral representations, Eq. (4) or (5), to explicitly compute the velocity values makes it feasible to confine the solution field to the vortical region of the mean flow.

Turbulence Model Equations

Equations (1-6) are obtained from differential equations for the instantaneous motion of the fluid through Reynolds decomposition. In principle then, the time-dependent equations for the instantaneous fluid motion apply equally to the turbulent flow and the laminar one. In practice, however, the important details of turbulence, being transient and small-scale in character, are not discernible in the time scale and the length scale appropriate to the mean flow. The development of reasonably accurate solution methods for the mean flow is at present, a more meaningful and rewarding goal in view of the incredibly large amount of computation necessary for the details of turbulent motion.

The mean flow equations invariably involve more unknowns than are appropriate to the equations. This well-known problem of closure is characteristic of all nonlinear stochastic systems. The underlying cause of this difficulty is again the involvement of scales vastly different in magnitude in the problem to be solved. This obstacle is less easy to remove, for, unlike the vortical and potential regions of the flow that physically occupy different parts of the flowfield, the small-scale fluctuating motion is an integral part of the vortical flow and coexists with the mean flow in the same vortical region. There is no prospect of separating the turbulent motion from the mean motion in the manner that the potential region is separated from the vortical region by the integro-differential formulation. Indeed, the majority of investigations of turbulent flow problems have been concerned with this closure problem, the most prominent and practical attempts being the construction of various turbulence models.

The term "turbulence model" implies assumptions, often ad hoc, which close the problem, i.e., make the number of unknowns appropriate to the number of equations and enable the simulation of the flow in its important aspects. The computation of turbulent flows based on any given model generally requires a knowledge of empirical constants. Thus a useful model must possess a degree of universality. That is, the model, with empirical constants deduced from a limited set of experimental conditions, must be expected to be capable of predicting flows under a wider set of experimental conditions. In the present study, a two-equation model, which has received a great deal of emphasis in recent years, is selected. This model is believed by many researchers to possess a reasonably wide scope of predictive ability for attached and separated flows. However, because of the obstacle of excessive computing needs discussed earlier, extensive testing and calibraton of this model have, only recently, begun for separated flows.

The current literature on turbulence contains a copious number of articles and models more sophisticated than the two-equation models. This first author has participated in the development of one such model and has studied others. The omission of these potentially more universal models from the present work is not due to mere carelessness, but results from the more advanced stage of development of the two-equation models and the relative economy with which two-equation models permit quantitative simulation of turbulent transport phenomena in separated flows. The focal point of the present work is to utilize the integro-differential formulation and develop a practical approach for studying separated mean turbulent flows. When more definite recommendations are available regarding the use of the more sophisticated models, it should be possible to incorporate them with the efforts described here, if necessary.

Consistent with this rationale, the selection of procedures for treating various aspects of the turbulence model is based mainly on the simplicity, computational economy, and well-establishedness of the options that are open. Several issues of immediate concern include the general form of a stress-strain relation to be used, the specific model to be selected among the several two-equation models available, and the procedure for accommodating the boundary regions where viscous forces are important. The relatively simple procedures described below are known to be satisfactory for the sample problems treated in the present work. The use of more universal procedures for problems involving more demanding circumstances is postponed.

The starting point of the two-equation model is the establishment of two transport equations for two field variables, e.g., the turbulent energy and a second variable, which may either be a turbulence length scale or a combination of the turbulent energy and the length scale. The transport equations are solved along with the equations for the mean flow. The computed distribution of these two variables are related to the Reynolds stress.

The several two-equation models in use today differ in their use of the two field variables. Several two-equation models have been extensively studied for boundary-layer type of flows and shown to provide prediction of the same level of accuracy. In the present work, the $k-\epsilon$ or turbulent kinetic energy-energy dissipation rate model is used. The considerable amounts of understanding and mathematical derivations that enabled the establishment of this model have been described by Launder et al.[18-20] The model equations used here are obtained from Ref. 18:

$$-\overline{v_i'v_j'}=\frac{2C_\mu k^2}{\epsilon}S_{ij} \tag{8}$$

$$\frac{\partial k}{\partial t}=-v_j\frac{\partial k}{\partial x_j}+\frac{\partial}{\partial x_j}\left(\frac{C_\mu k^2}{\epsilon}\frac{\partial k}{\partial x_j}\right)-\overline{v_i'v_j'}\frac{\partial v_i}{\partial x_j}-\epsilon \tag{9}$$

$$\frac{\partial \epsilon}{\partial t}=-v_j\frac{\partial \epsilon}{\partial x_j}+\frac{\partial}{\partial x_j}\left(\frac{C_\mu k^2}{\sigma_\epsilon \epsilon}\frac{\partial \epsilon}{\partial x_j}\right)-\frac{C_I\epsilon}{k}\overline{v_i'v_j'}\frac{\partial v_i}{\partial x_j}-\frac{C_2\epsilon^2}{k} \tag{10}$$

where the turbulent kinetic energy k and the dissipation rate ϵ are defined by

$$k=\tfrac{1}{2}\overline{v_j'v_j'} \tag{11}$$

and

$$\epsilon=\nu\overline{\left(\frac{\partial v_i'}{\partial x_j}\frac{\partial v_i'}{\partial x_j}\right)} \tag{12}$$

The numerical values of the constants in Eqs. (8-10) used in the present work are those recommended by Launder and Spalding[18]:

$$C_\mu=0.09 \quad C_I=1.44 \quad C_2=1.92 \text{ and } \sigma_\epsilon=1.3 \tag{13}$$

Equation (8) implies the notion of a scalar turbulent viscosity. Extensions of the $k-\epsilon$ model to account for more generally valid stress-strain relationships have been studied by others, but are not considered here. Equations (9) and (10) are valid for fully turbulent flows. It is possible to use a low Reynolds number modeling method,[10] and modify Eqs. (9) and (10) to accommodate these regions. In the present study, a simple procedure is adopted. The values of vorticity at data points away from solid boundaries are obtained by solving Eq. (6). The strength of vortex sheets on solid boundaries are then determined by placing these vorticity values into Eq. (5), requiring this equation to be satisfied on the solid boundaries where the velocity values are prescribed, and solving the resulting integral equation in the manner described in Ref. 14. The vortex sheet is considered to represent the total vorticity within a layer of fluid surrounding the surface of thickness Δn. With a known vortex sheet strength ζ at a particular boundary location, the following relations are used to determine first the friction velocity u_* at that location and then the values of ω, ϵ, and k at a data point p located at a normal distance n_p from the boundary location:

$$\zeta=-10u_*+\frac{u_*}{\kappa}\ln\left(\frac{10\nu}{u_*\Delta n}\right) \tag{14}$$

$$\omega_p=-u_*/\kappa n_p \tag{15}$$

$$\epsilon_p=u_*^3/\kappa n_p \tag{16}$$

$$k_p=u_*^2/\sqrt{C_\mu} \tag{17}$$

where κ is the von Karman constant taken as 0.41 and the friction velocity u_* is defined by

$$u_* = \left(\frac{\tau_0}{\rho}\right)^{1/2} \tag{18}$$

where τ_0 is the shear stress at the boundary location.

Relations (14-17) are based on the well-established notion for turbulent boundary layers that the surface layer near the solid boundary contains a viscous sublayer immediately adjacent to the solid boundary, where the Reynolds stress does not significantly contribute to the stress, and an inertial sublayer where the viscous stress is very small compared to the Reynolds stress, which is approximately constant in this sublayer. The vorticity in the viscous sublayer is constant since the velocity profile is linear there. The vorticity profile in the inertial layer, which corresponds to a logarithmic velocity profile, is inversely proportional to the normal distance from the solid boundary. The total vorticity within the surface layer is obtained by matching the velocity values given by the viscous and inertial sublayers at a normal distance $10\nu/u_*$ (Ref. 16). This simple procedure produces acceptable numerical results for problems involving relatively small pressure gradients, provided that the normal distances Δn and n_p are well within inertial sublayer.

Solution Procedure

The basic solution procedure adopted is as follows. The solution field is mapped by a data grid system. For convenience, data points on or immediately adjacent to the solid boundaries are called boundary points. Those away from the boundary and having nonzero vorticity are called vortical points. The following steps then constitute a computation loop to advance the solution by one time step:

1) Using known values of ω_i, k, ϵ, $-\overline{v_i'v_j'}$, and v_i at all boundary and vortical points at a given time level, say τ, Eqs. (6, 9, and 10) are solved numerically to determine new values of ω, k, and ϵ at a subsequent time level, say $\tau+\Delta t$, at all vortical points. Since the vortical region of the flow generally grows with time, the number of vortical points for the new time level is generally larger than that at the old time level. This involves three transport equations and constitutes the kinetic part of the computation.

2) The new ω_i values at the vortical points and Eq. (5) are utilized to establish the strength of boundary vortex sheets[14] at the new time level. This step is a part of the kinematic computation, as explained in Ref. 14.

3) Equations (14-17) are used to compute the new values of ω_i, k and ϵ at the boundary points. Equation (8) is used to compute $-\overline{v_i'v_j'}$ at all vortical points and boundary points. This step is part of turbulence modeling.

4) Using new values of ω_i at all vortical and boundary points, Eq. (5) or (4) is solved by numerical quadrature to obtain new values of v_i at the new time level. This step is the major part of the kinematic computation. The computation of v_i is performed only for the vortical points. The values of v_i at the boundary points are specified as boundary conditions. The values of v_i at data points where the vorticity is negligible, are not needed for the next computation loop and need not be computed.

The computation loop is initiated with prescribed values of ω_i, k, and ϵ at an initial time level at the vortical points only. These values determine the initial values of ω_i, k, and ϵ at the boundary points of $-\overline{v_i'v_j'}$ at all boundary and vortical points, and of v_i at all vortical points, as described by steps 2-4. In the present work, the fluid is considered to be initially at rest and is in contact with solid surfaces also initially at rest. The velocity of the solid surfaces rises suddenly at $t=0$ to some finite value and remains steady thereafter. The time level immediately after the start of the solid motion is taken to be the initial time level. Steady-state solutions are obtained asymptotically in the limit of large time. Since the flow is potential everywhere except on the solid boundaries at this initial time level, there are no vortical points initially. The procedure for the establishment of the strength of the boundary vortex sheets at the initial time level is equivalent to the procedure of Hess and Smith[21] for the establishment of source-sink distributions producing a potential flow. The initial strength of the vortex sheets lead to initial values of ω_i, k, and ϵ at the boundary points, through step 3. The initial values of ω_i, k, and ϵ are zero away from the solid boundaries.

Many options are available for the numerical procedures in each of the preceding steps. Equations (4) and (5) are well suited for the finite-element procedure which, like the coordinate transformation methods, permits the boundary geometry to be accurately represented by the data grid. The use of the finite-element procedure in conjunction with the integro-differential approach has been discussed in several previous articles.[8-10] The term "finite-element procedure" is used here in its broad sense to describe a numerical approach, in which the solution field is mapped into subregions, called elements, each associated with a finite number of data points called nodes. Continuum field variables are approximated in each element by specific functions and are expressed in terms

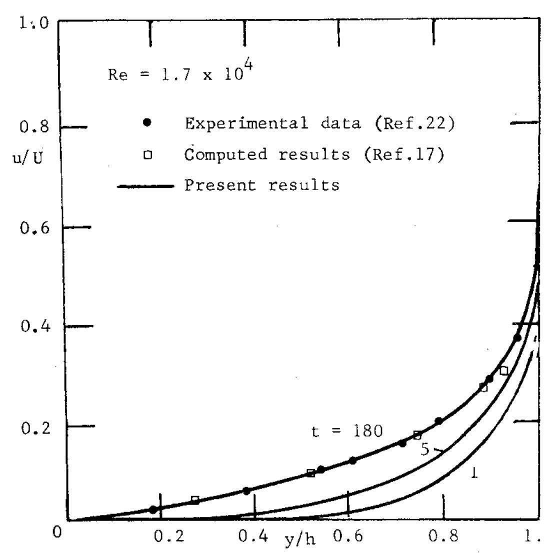

Fig. 1 Turbulent couette flow formation—mean velocity profile.

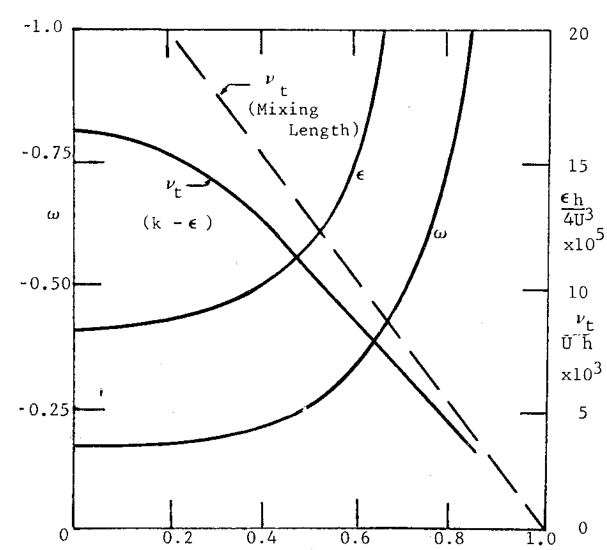

Fig. 2 Turbulent couette flow formation—profiles for mean vorticity energy dissipation rate and turbulent viscosity.

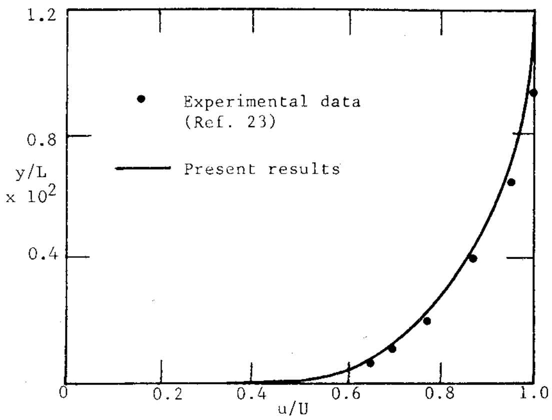

Fig. 3 Midplate mean velocity profile.

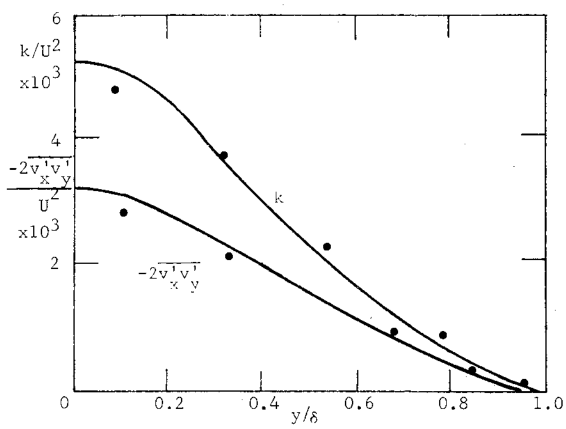

Fig. 4 Midplate turbulent kinetic energy and Reynolds stress.

of nodal values of the field variables. The field equations are then approximated accordingly by a set of algebraic equations containing nodal values of the field variables as unknowns. This procedure is particularly convenient for Eqs. (4) and (5). With this procedure, the integrals over the vortical region are replaced by sums of element integrals over individual elements. The latter integrals (members of the sum) are evaluated analytically and are expressed algebraically in terms of the coordinates of the nodes and the nodal values of vorticity. Analytical expressions have been obtained for any polygonal element and for any order of accuracy in representing the vorticity values. For the present work, since the problem treated involves simple boundary geometries, a simple numerical quadrature procedure[15] is used for evaluating the velocity values.

With a traditional finite-difference or finite-element method, the major portion of the required computation is spent in treating the kinematic equation. The integro-differential approach so drastically reduces the kinematic computation that the treatment of the kinetic equation now represents the major portion of the needed computation. The solution of the kinetic equations therefore replaces the solution of the kinematic equation as the limiting factor on computational speed for the overall problem. This limitation is more acute for turbulent flows than it is for laminar flows, since the former involve three kinetic equations while the latter involve only one. The development of highly efficient methods for solving kinetic equations is therefore highly important for turbulent flow studies. In this regard, a hybrid finite-element/finite-difference, implicit-explicit method[12] recently developed and tested for laminar flows is worthy of note. This method has been shown to lead to substantial reductions in computer time needed for the solution of the vorticity transport equation.

Results and Discussions

Several test problems involving relatively simple flow geometries have been studied using the integro-differential method in conjunction with the $k-\epsilon$ model. They include the problems of Couette flow formation, a sinusoidally oscillating mainstream past a finite plate, and a steady flow past a finite plate. In each of the problems, the motion of the fluid results from the impulsively started motion of the solid. The time-dependent solution for each problem is carried to a sufficiently large time level so that either a steady-state or periodic condition is reached. The primary purpose of solving these problems has been to verify the applicability of the integro-differential formulation in the computation of turbulent flows. Numerical procedures employed have been kept simple and unsophisticated by intention.

Figure 1 shows computed mean velocity profiles for the turbulent Couette flow formation problem. Two parallel infinite flat plates, separated by the distance $2h$, are initially at rest in the planes $y = \pm h$. The plates are set into motion, respectively, with the velocities $\pm U$ at the time level $t=0$ and thereafter kept at these velocities. The velocity profiles shown in Fig. 1 are for various time levels, nondimensionalized with respect to the time scale of $2h/U$, and for the Reynolds number Uh/ν of 17,000. The figures show that the computed velocity profile in the limit of large time is in excellent agreement with the experimental data of Reichardt,[22] as well as with numerical results obtained using a sophisticated "probability distribution function."[17] The computed steady-state profiles of the vorticity ω and the turbulent energy dissipation rate ϵ, shown in Fig. 2, exhibit the expected behavior of being roughly proportional to the inverse of the distance from the wall. The computed steady-state turbulent kinetic energy and Reynolds stress away from the surface layer, not shown here, are independent of the distance from the wall, as expected. The steady-state velocity profiles calculated using the mixing length theory not shown is in good agreement with Reichardt's data, as presented in Ref. 22. The turbulent viscosity profile based on the mixing length theory has a cusp at the midplate between the plates, which is not reasonable. This profile differs substantially from the turbulent viscosity profile deduced from the $k-\epsilon$ model, also shown in Fig. 2.

Figure 3 shows the mid-plate steady-state tangential velocity profile on a finite flat plate set impulsively into motion in its own plane in a direction perpendicular to its leading edge at the velocity U. The Reynolds number, based on U and the distance from the leading edge, is 2.66×10^6 at midplate. Figure 3 shows that the computed velocity profile is in good agreement with the experimental data of Smith and Walker[23] for a Reynolds number of 2.64×10^6. Figure 4 shows computed nondimensional turbulent kinetic energy and Reynolds stress profiles at midplate as well as Klebanoff's[24] experimental data. The computed results show the same trend as Klebanoff's data. Klebanoff used a scheme to thicken the boundary layer. His stated apparent Reynolds number, based on the distance from the virtual origin, is 4.2×10^6, which is significantly higher than the value used for the present computation. The local skin friction coefficient predicted in the present study, defined by $-2\overline{v_x' v_y'}/U^2$, is 3.07×10^{-3}. This value agrees extremely well with the value given by a semiempirical formula[22] for a Reynolds number based on the distance from the leading edge of 2.66×10^6. The measured skin friction value of Klebanoff, being at a higher Reynolds number, is, as expected, lower than the value computed here. The computed turbulent energy dissipation rates have the same qualitative agreement with Klebanoff's data as that exhibited by the turbulent kinetic energy and Reynolds stress profiles.

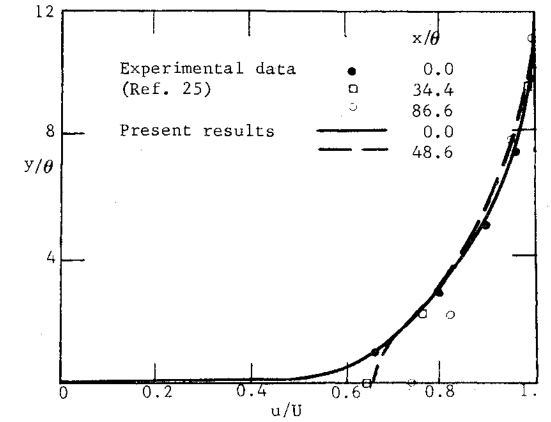

Fig. 5 Mean velocity profiles at the trailing edge and in the wake.

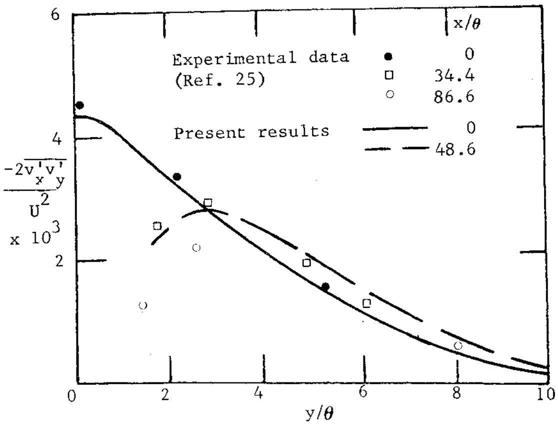

Fig. 6 Reynolds stress profiles at the trailing edge and in the wake.

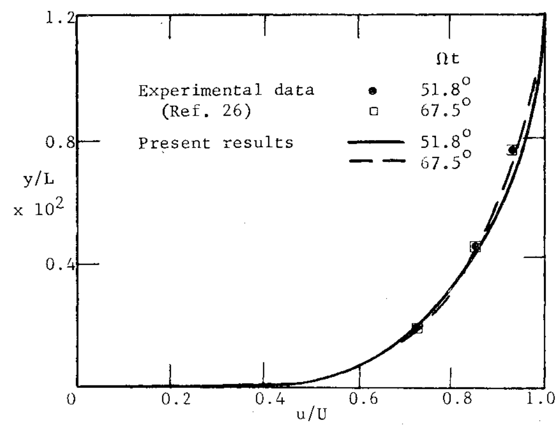

Fig. 7 Mean velocity profiles at midplate in an oscillating stream (low frequency).

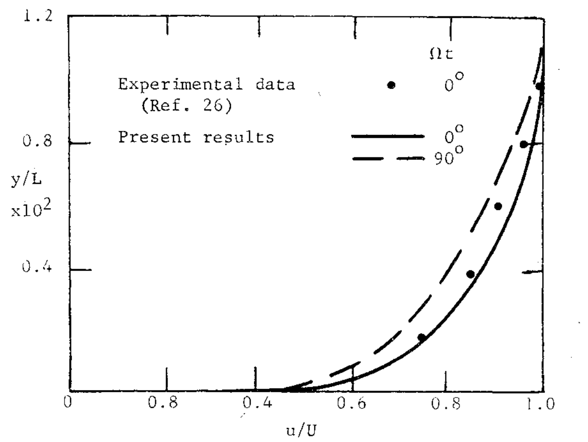

Fig. 8 Mean velocity profiles at midplate in an oscillating stream (high frequency).

The computed trailing-edge mean velocity profile is shown in Fig. 5 for a Reynolds number based on the plate length of 7×10^5. This Reynolds number was selected in order to compare the present results with the experimental data of Chevray and Kovasznay.[25] The comparison, shown in Fig. 5, indicates excellent agreement between the computed and measured velocity profiles at the trailing edge. Also shown in Fig. 5 are some computed and measured velocity profiles in the wake of the finite plate.

Figure 6 shows the computed Reynolds stress profiles at the trailing edge and in the wake, compared to the experimental data of Chevray and Kovasznay. The agreement between the computed results and the experimental data is good.

The comparisons between the computed results and experimental data previously described are all for steady-state conditions. At present, detailed information regarding time-dependent turbulent flows is scarce. Karlsson,[26] however, has obtained data for the flow of a sinusoidally fluctuating mainstream over a flat plate. In the present work, this problem is treated computationally. The results obtained are compared with Karlsson's data in Figs. 7 and 8. The mainstream velocity is considered to vary sinusoidally about an average value U_0 and is of the form

$$U=U_0(1+0.34\cos\Omega t) \tag{19}$$

where Ω is the nondimensional frequency with the reference time L/U_0, L being the plate length. The computation was performed for a low-frequency case with $\Omega=0.62\times2\pi$ and a high-frequency case with $\Omega=26\times2\pi$. The Reynolds number, based on the plate length and the average mainstream velocity, is 3×10^6, which gives an estimated midplate Reynolds number based on the displacement thickness of 3.6×10^3. This latter Reynolds number is that studied in Karlsson's experiments. Figure 7 shows velocity profiles for the low-frequency case at the mainstream phase angle Ωt of 51.8 and 67.5 deg. This figure shows that the flow is nearly "quasisteady" at this low frequency, as is expected.[27] That is, the mean velocity profiles, when nondimensionalized with respect to the mainstream velocity, are almost indistinguishable for different phase angles. The computed velocity profiles are in good agreement with Karlsson's data, which also show that the flow is quasisteady at this frequency. Note that the reference length L used to nondimensionalize Karlsson's data is the distance between a boundary-layer tripping device and the measurement station. The distance between the virtual origin and the measurement station should be somewhat greater than the reference length used to present Karlsson's data in Fig. 7. The computed profile and Karlsson's profile are in better agreement if the displacement thickness is used to nondimensionalize the results.

Figure 8 shows the computed velocity profiles for the high-frequency case at phase angles 0 and 90 deg. The results show that the flow is not quasisteady at this high frequency. Experimental data of Karlsson for this frequency are available for the phase angle of 0 deg only. These data are also shown in Fig. 8.

The computer time used to advance the solution for the flat-plate steady freestream problem by one unit of nondimensional time, which corresponds to the movement of the plate by one plate length relative to the freestream, is about 5 min on the CDC 6600 computer. As stated earlier, the numerical procedure used for the problems treated thus far has been kept simple. For the flat plate problems, a simple quadrature procedure is used for evaluating the integral in Eq. (5), and the successive over-relaxation method is used for treating the kinetic transport equations, Eqs. (6, 9, and 10). More sophisticated numerical procedures,[11,13] established earlier for the laminar flow problem, should offer substantial improvement in computational efficiency for turbulent flow. The experience with laminar flow problems indicates that the computer time needed for the flat plate problem in steady freestream can be reduced to less than one minute per unit nondimensional time without introducing adverse effects on the solution accuracy.

It was anticipated that the development of a routine computational capability for treating turbulent flow problems involving complex flow geometries would require extensive and persistent efforts over a number of years. The work described in this article represents only the few initial steps in search of this capability. In that context, the results of the present study are encouraging in that they lend support to the expectation that the integro-differential formulation is well-suited for the numerical solution of turbulent flow problems.

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